Decompositions of $\mathbb{R}^n, n \geq 4$, into convex sets generate codimension 1 manifold factors

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Abstract. We show that if G is an upper semicontinuous decomposition of $\mathbb{R}^n$, $n \geq 4$, into convex sets, then the quotient space $\mathbb{R}^n/G$ is a codimension one manifold factor. In particular, we show that $\mathbb{R}^n/G$ has the disjoint arc-disk property.

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1. Introduction

A space X is said to be a *codimension one manifold factor* provided that $X \times \mathbb{R}$ is a manifold. It is a long standing unsolved problem as to whether all resolvable generalized manifolds are codimension one manifold factors [9]. This is the so-called Product With a Line Problem and it is the essence of the famous Generalized R. L. Moore Problem [21, 25, 26].

The Product With a Line Problem speaks directly to one of the most fundamental questions in geometric topology, which is how to recognize manifolds [6, 15, 27, 28, 29]. Because manifolds have a rich structure which is useful to exploit in many areas of mathematics and its applications, it is important to recognize when one is dealing with a space that is a manifold. One notable example is the relevance of the Product With a Line Problem to the famous Busemann Conjecture in metric geometry [3, 4, 5, 19].

One might wonder even if a decomposition of $\mathbb{R}^n$ into convex sets could give rise to a decomposition space topologically distinct from $\mathbb{R}^n$. This problem was investigated for several years beginning with Bing in the 1950's [1, 2, 8, 14, 23]. In 1970, Armentrout [1] produced the first example of a decomposition of $\mathbb{R}^3$ into convex sets that yields a non-manifold. Then in 1975, Eaton [14] demonstrated that a certain decomposition of $\mathbb{R}^3$ into points and

straight line segments, originally proposed by Bing [2], is indeed topologically distinct from $\mathbb{R}^3$. Hence, this type of complexity is significant. It should also be noted that there are no known examples of a non-manifold resulting from a decomposition of $\mathbb{R}^{n\geq 4}$ into convex sets.

In this paper we shall investigate how the type of complexity represented by decompositions of $\mathbb{R}^n$ into convex sets can affect the classification of a decomposition space as a codimension one manifold factor. We shall demonstrate that decompositions of $\mathbb{R}^n$, $n \geq 4$, into convex sets are always codimension one manifold factors. In particular, we shall show that such spaces have a particularly strong general position property, the disjoint arc-disk property.

2. Preliminaries

We briefly review some basic definitions and notations. Recall that a map $f : X \to Y$ is said to be *proper* if whenever C is a compact subset of Y, then $f^{-1}(C)$ is compact.

There are various equivalent definitions of upper semicontinuous decompositions [11], but the following will be the most useful for our purposes:

Definition 2.1. A decomposition G of M into compact sets is said to be *upper semicontinuous* (usc) if and only if the associated decomposition map $\pi : M \to M/G$ is a proper map.

A compact subset C of a space X is said to be *cell-like* if for each neighborhood U of C in X, C can be contracted to a point inside U [24]. A usc decomposition G of M is said to be *cell-like* if each element $g \in G$ is cell-like. A map $f : Y \to X$ is said to be *cell-like* if for each $x \in X$, $f^{-1}(x)$ is cell-like. A *resolvable generalized n-manifold* is an n-dimensional space X that is the image of a cell-like map $f : M \to X$ where M is an n-manifold.

Convex sets are contractible, and hence they are cell-like. Thus, a usc decomposition G of $\mathbb{R}^n$ into convex sets is a cell-like decomposition and the associated decomposition map $\pi : \mathbb{R}^n \to \mathbb{R}^n/G$ is a cell-like map. The fact that $\mathbb{R}^n/G$ is finite-dimensional follows from a result of Zemke (see [30, Theorem 5.2]). Therefore, in this setting, $\mathbb{R}^n/G$ is a resolvable generalized n-manifold.

For resolvable generalized manifolds, we have the following very useful approximate lifting theorem, which follows from [11, Theorem 17.1 and Corollary 16.12B]:

Theorem 2.2. *Suppose that G is a cell-like decomposition of a manifold M, with decomposition map $\pi : M \to M/G$, and that the quotient space M/G is finite-dimensional. Then for any map $f : Z \to M/G$ of a finite-dimensional compact polyhedron Z, and any $\epsilon > 0$, there exists a map $F : Z \to M$ such that πF is an ϵ-approximation of f.*

General position properties are very useful in detecting codimension one manifold factors [12, 16, 17, 18, 20]. For our results, we shall only need to employ the following:

Definition 2.3. A space X is said to have the *disjoint arc-disk property* (DADP) provided that any two maps $\alpha : I \to X$ and $f : D^2 \to X$ can be approximated by maps with disjoint images, where I denotes the unit interval and D^2 denotes a disk.

The following theorem was demonstrated in [10, Proposition 2.10]:

Theorem 2.4. *A resolvable generalized manifold having DADP is a codimension one manifold factor.*

Useful in discussions of the DADP is the local 0-co-connectedness property. A set $Z \subset X$ is said to have the *local* 0*-co-connectedness property* (0-LCC) in X if for every $z \in Z \cap \mathrm{Cl}(X - Z)$, each neighborhood U of z contains another neighborhood V of z so that any two points in $V - Z$ are path connected in $U - Z$. Note that if Z is nowhere dense in X, then $Z = Z \cap \mathrm{Cl}(X - Z)$.

The following theorem can be found in [11, Corollary 26.2A]:

Theorem 2.5. *Each k-dimensional closed subset of a generalized n-manifold, where $k \le n - 2$, is* 0*-LCC.*

Since a k-dimensional closed subset of a generalized n-manifold X, where $k \le n - 1$, is nowhere dense in X, we have the following corollary:

Corollary 2.6. *If Z is a k-dimensional closed subset of a generalized n-manifold X, where $k \le n - 2$, then any path $\alpha : I \to X$ can be approximated by a path $\alpha' : I \to X - Z$.*

3. Main Results

The main result of this paper is the following theorem:

Theorem 3.1. *Let G be an upper semicontinuous decomposition of $\mathbb{R}^n$ into convex sets, where $n \ge 4$. Then $\mathbb{R}^n/G$ is a codimension one manifold factor.*

This theorem will follow immediately as a corollary of Theorem 2.4 and the following theorem:

Theorem 3.2. *Let G be an upper semicontinuous decomposition of $\mathbb{R}^n$ into convex sets, where $n \ge 4$. Then $\mathbb{R}^n/G$ has the DADP.*

Proof. Let $f : D^2 \to \mathbb{R}^n/G$ and $\varepsilon > 0$. It follows from Corollary 2.6 that it suffices to show that there is an ϵ-approximation $f' : D^2 \to \mathbb{R}^n/G$ of f such that $f'(D^2)$ is 2-dimensional.

Let $F : D^2 \to \mathbb{R}^n$ be a piecewise linear map, that is an ε-approximate lift of f. We shall show that $f' = \pi F$ is then the desired map.

Let T denote a triangulation of $F(D^2)$. We claim that if σ is a 2-simplex of T, then $f'(\sigma)$ is 2-dimensional. To see this, let G_σ be the decomposition induced over $\pi(\sigma)$, i.e. G_σ is the decomposition of $\mathbb{R}^n$ having as the only nontrivial elements, the nontrivial elements of G that meet σ. Let $\omega : \mathbb{R}^n \to \mathbb{R}^n/G_\sigma$ be the associated decomposition map. Note that ω is necessarily a proper

map, being a decomposition induced over a closed set in the decomposition space of a usc decomposition.

Let P be the 2-dimensional plane in $\mathbb{R}^n$ that contains σ. Let ϖ denote the restriction of ω to P. Then ϖ is also a proper map. Thus ϖ determines a usc decomposition of the plane into convex sets, elements that do not separate the plane. It now follows from a classical result of Moore [25, 26], that ϖ is a near-homeomorphism onto its image. Thus $\varpi(\sigma)$ is at most 2-dimensional.

But $\varpi(\sigma)$ is homeomorphic to $\omega(\sigma)$, which in turn is homeomorphic to $\pi(\sigma)$. Thus $\pi(\sigma)$ is at most 2-dimensional subset of $\mathbb{R}^n/G$. Hence

$$f'(D^2) = \bigcup_{\sigma \in T^{(2)}} \pi(\sigma)$$

is a 2-dimensional subset of the generalized n-manifold $\mathbb{R}^n/G$ [22]. □

4. Conclusions

As we have seen, the complexity represented by decompositions into convex sets does not inhibit a decomposition space from being a codimension one manifold factor. The fact that such spaces satisfy the DADP is a pleasant result.

It is well known that not all codimension one manifold factors satisfy the DADP, and hence the DADP is not a general position property that provides a characterization of codimension one manifold factors. In fact, the DADP condition is a relatively weak tool for detecting codimension one manifold factors, compared to other general position properties such as:

- the disjoint homotopies property [16];
- the plentiful 2-manifolds property [16];
- the 0-stitched disks property [18];
- the method of δ-fractured maps [17]; and
- the disjoint topographies (or disjoint concordance) property [12, 20].

It is these stronger properties that must be utilized to demonstrate that spaces such as the Totally Wild Flow [7] and the Ghastly Spaces [13] are codimension one manifold factors.

In conclusion, we have demonstrated that we must look to other types of complexities to realize a counterexample to the Generalized R. L. Moore Problem, if such an example does indeed exist.

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